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Analytic solutions to Maxwell–London equations and levitation force for a general magnetic source in the presence of a long type-II superconducting cylinder

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Abstract. The interaction between a general magnetic source and a long type-II superconducting cylinder in the Meissner or mixed state is studied within the London theory. We first study the Meissner state and solve the Maxwell–London equations when the source is a magnetic monopole located at an arbitrary position. Then the field and supercurrent for a more complicated magnetic charge distribution can be obtained by superposition. A magnetic point dipole with arbitrary direction is studied in detail. It turns out that the levitation force on the dipole contains in general an angular as well as a radial component. By integration we obtain the field and supercurrent when the source is a two-dimensional monopole (a magnetically charged long thread along the axial direction), from which the results for a two-dimensional point dipole easily follow. In the latter case the levitation force points in the radial direction regardless of the orientation of the dipole. The case for a current carrying long straight wire parallel to the cylindrical axis is solved separately. The limit of ideal Meissner state is discussed in most cases. The case of mixed state is discussed briefly. It turns out that vortex lines along the axial direction and vortex rings concentric with the cylinder have no effect outside the cylinder and the levitation forces remain the same as in the case of the Meissner state.

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1 Introduction

The magnetic behavior of macroscopic superconductors in the Meissner or mixed state can be approximately described by the phenomenological London theory. Physical problems involving magnetic sources and superconductors are reduced to magnetostatic boundary value problems in this approximation. More specifically, the magnetic induction and supercurrent distribution can be found by solving the Maxwell–London equations with appropriate boundary conditions. From these solutions one can calculate the levitation force between the magnetic source and the superconductor. Magnetic levitation is a characteristic property of superconductors [1]. It has been argued that the temperature dependence of the London penetration depth and thus that of the magnetic levitation force is important in studying the superconducting paring state [2]. This relates the macroscopic phenomenological study to the microscopic mechanism of superconductivity.

The above magnetostatic boundary value problems can be solved analytically when the boundaries have simple geometric configurations. The simplest boundaries are plane, sphere and cylinder. Many works have been devoted to the various cases with planar boundary conditions [2–8]. The case with a spherical boundary has also been studied by many authors [9–14]. On the other hand, the case with a cylindrical boundary appears to be less studied. Vortex lines in a long superconducting cylinder induced by external fields [15] or vortex rings induced by the self-field of a transport current [16] have been considered. These fields are homogeneous along the axial direction of the cylinder. The behavior of a finite superconducting cylinder under a cylindrically symmetric external field were also studied in the literature [17] (by numerical method). Recently some magnetostatic boundary value problems involving a long superconducting cylinder in the ideal Meissner state was treated by the image method [13], where the magnetic sources are uniform along the axial direction. However, the most typical case of a long superconducting cylinder

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under the influence of a general external magnetic source (a point dipole, say) appears to be unsolved so far. The situation is somewhat unexpected since the London theory has been developed for a rather long time. The external field in this case has no symmetry, and the vector nature of the Maxwell–London equations leads to some mathematical difficulty. However, these equations are similar to the Maxwell equations for monochromatic fields in free space, and the approach to the solutions of these Maxwell equations under cylindrical boundary is known [18]. The approach can be appropriately modified to derive the solutions for the current problem. The purpose of this paper is to fill this gap. This may be of some interest in the study of magnetic levitation [1,2,4–13,17,19–25] and may have some applications to magnetic force microscopy [26–29].

In Section 2 we first study the general approach to the solution of the Maxwell–London equations for the system of a general magnetic source and a long superconducting cylinder in the Meissner state. Then we present the elementary solution for the simplest case where the source is a magnetic monopole. The fields and supercurrent for more complicated distributions of magnetic charges (a point dipole, say) can in principle be obtained from the elementary solution by superposition. A magnetic point dipole with arbitrary direction is studied in detail in Section 3. It turns out that the levitation force on the magnetic dipole contains in general an angular as well as a radial component. In Section 4 we consider some cases where the magnetic sources are uniform along the axial direction, so that they can be treated as two-dimensional problems. These include the cases of a two-dimensional magnetic monopole (a magnetically charged long thread parallel to the axis of the cylinder), a two-dimensional point dipole, a uniform magnetic field perpendicular to the axis of the cylinder, and a current carrying long straight wire parallel to the axis of the cylinder. The solutions in all these cases are obtained from the elementary solution by superposition except for the final one which is solved individually. The levitation force between the superconducting cylinder and the magnetic source is calculated in all cases. It turns out that the levitation force on a two-dimensional point dipole is independent of the dipole's orientation. In Section 5 we briefly discuss the mixed state of the superconducting cylinder. It was already known that vortex lines parallel to the axis of the cylinder and vortex rings concentric with the cylinder have no field outside the cylinder. Therefore the levitation forces remain the same as those obtained for the Meissner state. Section 6 is devoted to a brief summary and some discussions. Throughout this paper we use MKS units.

2 General formalism and the elementary solution

2.1 General formalism

Consider a general magnetic source in the presence of a long superconducting cylinder in the Meissner state. We choose the coordinate system such that the z axis coincides with the axis of the cylinder. We will use the rectangular coordinates (x, y, z) as well as the cylindrical ones (ρ, ϕ, z). The unit vectors in these coordinate systems are denoted by $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ and $(\boldsymbol{e}_\rho, \boldsymbol{e}_\phi, \boldsymbol{e}_z)$, respectively. The position vector is denoted by $\boldsymbol{r}$ and the one on the xy plane is denoted by $\boldsymbol{\rho}$. The magnetic induction outside the cylinder is denoted by $\boldsymbol{B}_1$ and the one inside it by $\boldsymbol{B}_2$. The supercurrent density inside the cylinder is denoted by $\boldsymbol{J}$. We assume that the superconducting cylinder is insulated to external current sources so that the total supercurrent along the z direction vanishes, i.e., $\int_{\rho<a} J_z \, d\boldsymbol{\rho} = 0$. The cylinder has radius a, and in the subsequent discussions it will be idealized as infinitely long.

The magnetic induction outside the cylinder can be decomposed as

$$\boldsymbol{B}_1 = \boldsymbol{B}_0 + \boldsymbol{B}', \tag{1}$$

where $\boldsymbol{B}_0$ is generated by the given source, which can be written down directly, and $\boldsymbol{B}'$ is generated by the induced supercurrent in the cylinder. The Maxwell equation for $\boldsymbol{B}'$ is obviously

$$\nabla \cdot \boldsymbol{B}' = 0, \quad \nabla \times \boldsymbol{B}' = 0. \tag{2}$$

The magnetic induction $\boldsymbol{B}_2$ and the supercurrent $\boldsymbol{J}$ inside the cylinder satisfy the Maxwell–London equation

$$\nabla \cdot \boldsymbol{B}_2 = 0, \quad \nabla \cdot \boldsymbol{J} = 0, \tag{3a}$$

$$\nabla \times \boldsymbol{B}_2 = \mu_0 \boldsymbol{J}, \quad \nabla \times \boldsymbol{J} = -\frac{\kappa^2}{\mu_0} \boldsymbol{B}_2, \tag{3b}$$

where κ is a phenomenological parameter with $1/\kappa$ being the London penetration depth. The boundary condition at the surface of the cylinder is

$$\boldsymbol{B}_1|_{\rho=a} = \boldsymbol{B}_2|_{\rho=a}. \tag{4}$$

We assume that there is no electric current when $\rho \gtrsim a$, then in that region we have $\nabla \times \boldsymbol{B}_1 = 0$. Using the first equation in equation (3b) (Ampere's law) and the above boundary condition we have

$$J_\rho|_{\rho=a} = 0. \tag{5}$$

A correct solution should satisfy this condition. It means that the supercurrent is confined to the cylinder, which is physically obvious.

Though the region outside the cylinder is not simply connected, we can still introduce a magnetic scalar potential φ' such that $\boldsymbol{B}' = -\mu_0 \nabla \varphi'$, since the total supercurrent in the z direction vanishes. Then φ' satisfies the Laplace equation and the solution regular at infinity is

$$\varphi'(\boldsymbol{r}) = \int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} f_m(k) K_m(|k|\rho) e^{im\phi} e^{ikz}, \quad \rho \geq a, \tag{6}$$

where $K_m(|k|\rho)$ are Bessel functions of imaginary argument, and the coefficients $f_m(k)$ are to be determined below.

The Maxwell–London equations (3) are similar to the Maxwell equations for monochromatic fields in free space. The approach to the solutions of the latter under cylindrical boundary conditions [18] can be appropriately modified to derive the solutions of the former. It is easy to show that both $\boldsymbol{B}_2$ and $\boldsymbol{J}$ satisfy the vector Helmholtz equation:

$$\nabla^2 \boldsymbol{B}_2 - \kappa^2 \boldsymbol{B}_2 = 0, \quad \nabla^2 \boldsymbol{J} - \kappa^2 \boldsymbol{J} = 0. \tag{7}$$

In particular, we have

$$\nabla^2 B_{2z} - \kappa^2 B_{2z} = 0, \quad \nabla^2 J_z - \kappa^2 J_z = 0. \tag{8}$$

The solutions regular at $\rho = 0$ are easily found to be

$$B_{2z}(\boldsymbol{r}) = \int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} a_m(k)\gamma_k^2 I_m(\gamma_k\rho) e^{im\phi} e^{ikz}, \ \rho \le a, \tag{9a}$$

$$J_z(\boldsymbol{r}) = \int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} b_m(k)\gamma_k^2 I_m(\gamma_k\rho) e^{im\phi} e^{ikz}, \ \rho \le a, \tag{9b}$$

where $\gamma_k = \sqrt{k^2 + \kappa^2}$ (the factor γ_k^2 is introduced here for the following convenience), $I_m(\gamma_k\rho)$ are Bessel functions of imaginary argument, and the coefficients $a_m(k)$ and $b_m(k)$, together with $f_m(k)$ above, are to be determined by the boundary condition (4). We define the transverse vectors and operator as $\boldsymbol{B}_{2t} = \boldsymbol{B}_2 - B_{2z}\boldsymbol{e}_z$, $\boldsymbol{J}_t = \boldsymbol{J} - J_z\boldsymbol{e}_z$, and $\nabla_t = \nabla - \boldsymbol{e}_z\partial_z$. For a single Fourier component in z, that is, when B_{2z} and J_z is proportional to e^{ikz}, equation (3b) can be solved to give

$$\boldsymbol{B}_{2t}(\boldsymbol{r}) = -i\frac{k}{\gamma_k^2}\nabla_t B_{2z}(\boldsymbol{r}) + \frac{\mu_0}{\gamma_k^2}\boldsymbol{e}_z \times \nabla_t J_z(\boldsymbol{r}), \tag{10a}$$

$$\boldsymbol{J}_t(\boldsymbol{r}) = -i\frac{k}{\gamma_k^2}\nabla_t J_z(\boldsymbol{r}) - \frac{\kappa^2}{\mu_0\gamma_k^2}\boldsymbol{e}_z \times \nabla_t B_{2z}(\boldsymbol{r}). \tag{10b}$$

It can be shown that equation (3a) is also satisfied by these solutions. According to equations (9) and (10), we have

$$\boldsymbol{B}_{2t}(\boldsymbol{r}) = -i\nabla_t \int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} a_m(k) k I_m(\gamma_k\rho) e^{im\phi} e^{ikz}$$
$$+\mu_0 \boldsymbol{e}_z \times \nabla_t \int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} b_m(k) I_m(\gamma_k\rho) e^{im\phi} e^{ikz}, \ \rho \le a, \tag{11a}$$

$$\boldsymbol{J}_t(\boldsymbol{r}) = -i\nabla_t \int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} b_m(k) k I_m(\gamma_k\rho) e^{im\phi} e^{ikz}$$
$$-\frac{\kappa^2}{\mu_0}\boldsymbol{e}_z \times \nabla_t \int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} a_m(k) I_m(\gamma_k\rho) e^{im\phi} e^{ikz}, \ \rho \le a. \tag{11b}$$

When the source is given we can write $\boldsymbol{B}_0$ in similar forms to the above solutions, and work out the unknown coefficients by the boundary condition (4). In the following we will first find the solution when the source is a monopole located at an arbitrary position (the elementary solution), and then obtain the solutions for more complicated charge distributions by superposition.

2.2 The elementary solution

First we consider the simplest case of a magnetic monopole with magnetic charge Q located at the position $\boldsymbol{r}_0 = (\rho_0, \phi_0, z_0)$ (the position is always given in cylindrical coordinates) where $\rho_0 > a$. Though a magnetic monopole has not been observed in experiment and thus is artificial, the result is important for the subsequent problems.

The magnetic induction of the monopole can be expressed in terms of a magnetic scalar potential: $\boldsymbol{B}_0 = -\mu_0\nabla\varphi_0$, where

$$\varphi_0(\boldsymbol{r}) = \frac{Q}{4\pi|\boldsymbol{r} - \boldsymbol{r}_0|}. \tag{12a}$$

This can be expanded near the surface of the cylinder as

$$\varphi_0(\boldsymbol{r}) = \frac{Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} K_m(|k|\rho_0)$$
$$\times I_m(|k|\rho) e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho < \rho_0. \tag{12b}$$

According to the approach presented in the preceding section, we obtain the following elementary solution

$$\varphi'(\boldsymbol{r}) = \frac{Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} g_m(k) K_m(|k|\rho_0)$$
$$\times K_m(|k|\rho) e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \ge a. \tag{13}$$

$$B_{2z}(\boldsymbol{r}) = -i\frac{\mu_0 Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} \gamma_k^2 c_m(k) K_m(|k|\rho_0)$$
$$\times I_m(\gamma_k\rho) e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \le a, \tag{14a}$$

$$B_{2\rho}(\boldsymbol{r}) = -\frac{\mu_0 Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} \gamma_k K_m(|k|\rho_0)$$
$$\times \left[k\, c_m(k) I_m'(\gamma_k\rho) + m\kappa^2 d_m(k)\frac{I_m(\gamma_k\rho)}{\gamma_k\rho}\right]$$
$$\times e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \le a, \tag{14b}$$

$$B_{2\phi}(\boldsymbol{r}) = -i\frac{\mu_0 Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} \gamma_k K_m(|k|\rho_0) \times \left[mk\, c_m(k)\frac{I_m(\gamma_k\rho)}{\gamma_k\rho} + \kappa^2 d_m(k) I'_m(\gamma_k\rho)\right] \times e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \leq a. \tag{14c}$$

$$J_z(\boldsymbol{r}) = -i\frac{\kappa^2 Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} \gamma_k^2 d_m(k) K_m(|k|\rho_0) \times I_m(\gamma_k\rho) e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \leq a, \tag{15a}$$

$$J_\rho(\boldsymbol{r}) = \frac{\kappa^2 Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} \gamma_k K_m(|k|\rho_0) \times \left[m\, c_m(k)\frac{I_m(\gamma_k\rho)}{\gamma_k\rho} - k\, d_m(k) I'_m(\gamma_k\rho)\right] \times e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \leq a, \tag{15b}$$

$$J_\phi(\boldsymbol{r}) = i\frac{\kappa^2 Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} \gamma_k K_m(|k|\rho_0) \times \left[c_m(k) I'_m(\gamma_k\rho) - mk\, d_m(k)\frac{I_m(\gamma_k\rho)}{\gamma_k\rho}\right] \times e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \leq a. \tag{15c}$$

Here the prime indicates differentiation with respect to the argument. The coefficients are given by

$$g_m(k) = \frac{g_{1m}(k)}{g_{2m}(k)},$$
$$c_m(k) = \frac{k\gamma_k a I'_m(\gamma_k a)}{g_{2m}(k)},$$
$$d_m(k) = \frac{m I_m(\gamma_k a)}{g_{2m}(k)}, \tag{16a}$$

and

$$g_{1m}(k) = -I_m(|k|a)[k^2 a^2 \gamma_k^2 I'^2_m(\gamma_k a) + m^2\kappa^2 I_m^2(\gamma_k a)] + |k| a^2 \gamma_k^3 I'_m(|k|a) I'_m(\gamma_k a) I_m(\gamma_k a), \tag{16b}$$

$$g_{2m}(k) = K_m(|k|a)[k^2 a^2 \gamma_k^2 I'^2_m(\gamma_k a) + m^2\kappa^2 I_m^2(\gamma_k a)] - |k| a^2 \gamma_k^3 K'_m(|k|a) I'_m(\gamma_k a) I_m(\gamma_k a). \tag{16c}$$

Some remarks on the solution. First, all coefficients are independent of $\boldsymbol{r}_0$ (the dependence of the above solution on $\boldsymbol{r}_0$ has been explicitly factored out). This is convenient for subsequent applications to more complicated sources. Second, it is easy to see that $g_m(k)$ is even in both k and m; $c_m(k)$ is even in m and odd in k; while $d_m(k)$ is even in k and odd in m. Then the solution can be easily written in an explicitly real form. However, the above form is more compact and more convenient for the calculations in subsequent sections. Third, it can be verified that equation (5) is indeed satisfied. Fourth, the solution is rather complicated. Let us check some limit cases.

(1) $\kappa \to 0$. This means that the London penetration depth is infinitely large. In other words, the superconducting cylinder is absent. Careful analysis of the above results yields in this limit $\varphi' = 0$, $\boldsymbol{J} = 0$, and $\boldsymbol{B}_2 = -\mu_0\nabla\varphi_0 = \boldsymbol{B}_0$. These are physically expected results.
(2) $a \to 0$. Again this is the limit of a vanishing superconducting cylinder. In this limit the field inside the cylinder can be ignored. Careful analysis of the above results yields again $\varphi' = 0$, just as expected. One should also ensure that $c_m(k)$ and $d_m(k)$ do not become singular when $a \to 0$. This can be verified.
(3) $\kappa \to \infty$. This is the limit of ideal Meissner state. In this case it can be shown that

$$\varphi'(\boldsymbol{r}) = -\frac{Q}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} \frac{I'_m(|k|a)}{K'_m(|k|a)} \times K_m(|k|\rho_0) K_m(|k|\rho) e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \geq a. \tag{17}$$

Then it turns out that $\partial\varphi_1/\partial\rho|_{\rho=a} = 0$, which is the boundary condition for the ideal Meissner state. It can also be shown that B_{2z} and $B_{2\phi}$ vanish for $\rho < a$ while $B_{2\rho}$ vanishes for $\rho \leq a$, all are expected. For the supercurrent, we have $J_\rho = 0$ for $\rho \leq a$, and

$$J_z(\boldsymbol{r}) = i\frac{Q}{4\pi^2 a}\delta(\rho - a) \times \int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} \frac{m K_m(|k|\rho_0)}{|k| a K'_m(|k|a)} \times e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \leq a, \tag{18a}$$

$$J_\phi(\boldsymbol{r}) = -i\frac{Q}{4\pi^2 a}\delta(\rho - a) \times \int_{-\infty}^{+\infty} dk\, \epsilon(k) \sum_{m=-\infty}^{+\infty} \frac{K_m(|k|\rho_0)}{K'_m(|k|a)} \times e^{im(\phi-\phi_0)} e^{ik(z-z_0)}, \quad \rho \leq a, \tag{18b}$$

where $\epsilon(k) = k/|k|$ is the sign function. In this case they are surface currents. The surface current density can be worked out by the relations $K_z = (1/a)\int_0^a J_z\rho\, d\rho$ and $K_\phi = \int_0^a J_\phi\, d\rho$. Then it can be shown that $\boldsymbol{e}_\rho \times \boldsymbol{H}_1 = \boldsymbol{K}$, where $\boldsymbol{H}_1 = \boldsymbol{B}_1/\mu_0$. This is another boundary condition for the ideal Meissner state. Thus all results are expected ones.

Now we calculate the levitation force on the magnetic monopole. It is easy to find that

$$\boldsymbol{B}'(\boldsymbol{r}_0) = -\boldsymbol{e}_{\rho 0}\frac{\mu_0 Q}{2\pi^2}\int_0^\infty dk\, k\left[g_0(k)K_0(k\rho_0)K_0'(k\rho_0) + 2\sum_{m=1}^{\infty} g_m(k)K_m(k\rho_0)K_m'(k\rho_0)\right], \quad (19)$$

where $\boldsymbol{e}_{\rho 0} = \boldsymbol{e}_\rho(\boldsymbol{r}_0)$ (note that $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$ are functions of the position). Therefore the levitation force is

$$\boldsymbol{F} = Q\boldsymbol{B}'(\boldsymbol{r}_0) = -\boldsymbol{e}_{\rho 0}\frac{\mu_0 Q^2}{2\pi^2} \times \int_0^\infty dk\, k\left[g_0(k)K_0(k\rho_0)K_0'(k\rho_0) + 2\sum_{m=1}^{\infty} g_m(k)K_m(k\rho_0)K_m'(k\rho_0)\right]. \quad (20)$$

It points in the radial direction and the magnitude is independent of ϕ_0 and z_0, as is expected on the basis of the geometric symmetry of the system. In the limit of ideal Meissner state, this reduces to

$$\boldsymbol{F} = \boldsymbol{e}_{\rho 0}\frac{\mu_0 Q^2}{2\pi^2}\int_0^\infty dk\, k\left[\frac{I_0'(ka)}{K_0'(ka)}K_0(k\rho_0)K_0'(k\rho_0) + 2\sum_{m=1}^{\infty}\frac{I_m'(ka)}{K_m'(ka)}K_m(k\rho_0)K_m'(k\rho_0)\right]. \quad (21)$$

When $d = \rho_0 - a \ll a$, the superconducting cylinder can be approximately treated as an infinite superconducting plane, so that the above result should further reduce to $\boldsymbol{F} = \boldsymbol{e}_{\rho 0}(\mu_0 Q^2/16\pi d^2)$. However, we still do not know how to obtain this result from the above one since the summation is rather complicated. This is an open question.

To conclude this section we briefly discuss the effect of a nonvanishing supercurrent along the z direction. If the superconducting cylinder is connected to some external current source such that $\int_{\rho<a} J_z\, d\boldsymbol{\rho} = I_z$ is not zero, then we should introduce the magnetic scalar potential φ' through $\boldsymbol{B}' = \boldsymbol{e}_\phi(\mu_0 I_z/2\pi\rho) - \mu_0\nabla\varphi'$. By similar calculations to the above we find that $B_{2\phi}$ in equation (14c) gains an additional term $\mu_0 I_z I_1(\kappa\rho)/2\pi a I_1(\kappa a)$ and J_z in equation (15a) gains $\kappa I_z I_0(\kappa\rho)/2\pi a I_1(\kappa a)$, while all other results remain the same. Therefore the effect of I_z outside the cylinder is the same as that of a transport current along the z axis. Since this is familiar we will set $I_z = 0$ throughout this paper.

3 Magnetic point dipole

In this section we study the case where the source is a magnetic point dipole with dipole moment $\boldsymbol{m}_0$ located at the arbitrary position $\boldsymbol{r}_0$. This is of physical interest since it can be realized by a uniformly magnetized sphere of permeable material. The solution can be obtained by using the elementary solution and superposition. This is a typical application of the elementary solution.

Theoretically the above point dipole can be realized by two monopoles, one with charge $-Q$ located at $\boldsymbol{r}_0$ and the other with charge Q located at $\boldsymbol{r}_0 + \boldsymbol{l}$, where $\boldsymbol{l} \to 0$ and $Q \to \infty$ but $Q\boldsymbol{l} = \boldsymbol{m}_0$ is fixed. Since $\boldsymbol{l} \to 0$ the cylindrical coordinates of $\boldsymbol{r}_0 + \boldsymbol{l}$ is $(\rho_0 + \delta\rho, \phi_0 + \delta\phi, z_0 + \delta z)$ where $\delta\rho = \boldsymbol{l}\cdot\boldsymbol{e}_{\rho 0}$, $\delta\phi = \boldsymbol{l}\cdot\boldsymbol{e}_{\phi 0}/\rho_0$, $\delta z = \boldsymbol{l}\cdot\boldsymbol{e}_z$, and $\boldsymbol{e}_{\phi 0} = \boldsymbol{e}_\phi(\boldsymbol{r}_0)$. The magnetic scalar potential of this source dipole is

$$\varphi_0(\boldsymbol{r}) = \frac{\boldsymbol{m}_0\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|^3}. \quad (22)$$

This is a result of the superposition of two monopole potentials of the form (12a). By superposition of two elementary solutions we obtain the following solution for the present case.

$$\begin{aligned}\varphi'(\boldsymbol{r}) = {} & \frac{\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0}}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} |k| g_m(k) \times K_m'(|k|\rho_0)K_m(|k|\rho)e^{im(\phi-\phi_0)}e^{ik(z-z_0)} \\ & - i\frac{\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0}}{4\pi^2\rho_0}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} m g_m(k) \times K_m(|k|\rho_0)K_m(|k|\rho)e^{im(\phi-\phi_0)}e^{ik(z-z_0)} \\ & - i\frac{\boldsymbol{m}_0\cdot\boldsymbol{e}_z}{4\pi^2}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty} k g_m(k) \times K_m(|k|\rho_0)K_m(|k|\rho)e^{im(\phi-\phi_0)}e^{ik(z-z_0)}, \quad \rho \ge a. \end{aligned} \quad (23)$$

$$B_{2z}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi^2\rho_0}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty}\left[i(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0})|k|\rho_0 \times K_m'(|k|\rho_0) + (\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0})mK_m(|k|\rho_0) + (\boldsymbol{m}_0\cdot\boldsymbol{e}_z)k\rho_0 K_m(|k|\rho_0)\right]\gamma_k^2 c_m(k) \times I_m(\gamma_k\rho)e^{im(\phi-\phi_0)}e^{ik(z-z_0)}, \quad \rho \le a, \quad (24a)$$

$$J_z(\boldsymbol{r}) = -\frac{\kappa^2}{4\pi^2\rho_0}\int_{-\infty}^{+\infty} dk \sum_{m=-\infty}^{+\infty}\left[i(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0})|k|\rho_0 \times K_m'(|k|\rho_0) + (\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0})mK_m(|k|\rho_0) + (\boldsymbol{m}_0\cdot\boldsymbol{e}_z)k\rho_0 K_m(|k|\rho_0)\right]\gamma_k^2 d_m(k) \times I_m(\gamma_k\rho)e^{im(\phi-\phi_0)}e^{ik(z-z_0)}, \quad \rho \le a. \quad (24b)$$

The other components $\boldsymbol{B}_{2t}$ and $\boldsymbol{J}_t$ can be obtained in the same way, or by using the above result and equation (11). Since these are not needed below we do not give them here.

Now we calculate the levitation force on the magnetic dipole. It can be calculated by the following formula [18].

$$\boldsymbol{F} = \nabla(\boldsymbol{m}_0\cdot\boldsymbol{B}')|_{\boldsymbol{r}=\boldsymbol{r}_0} = -\mu_0\nabla(\boldsymbol{m}_0\cdot\nabla\varphi')|_{\boldsymbol{r}=\boldsymbol{r}_0}, \quad (25)$$

where $\boldsymbol{r}$ is replaced by $\boldsymbol{r}_0$ only after all differentiations are carried out. We have

$$\begin{aligned}\boldsymbol{m}_0\cdot\nabla\varphi' = {} & [(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0})\cos(\phi-\phi_0) \\ & +(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0})\sin(\phi-\phi_0)]\partial_\rho\varphi' \\ & +\frac{1}{\rho}[(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0})\cos(\phi-\phi_0) \\ & -(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0})\sin(\phi-\phi_0)]\partial_\phi\varphi'+(\boldsymbol{m}_0\cdot\boldsymbol{e}_z)\partial_z\varphi'. \end{aligned}\tag{26}$$

Substituting this into equation (25), using equation (23), working out the differentiations and finally set $\boldsymbol{r}=\boldsymbol{r}_0$, we obtain

$$\begin{aligned}F_\rho = {} & -\frac{\mu_0(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0})^2}{2\pi^2}\int_0^\infty dk\,k^3[g_0(k)K_0'(k\rho_0) \\ & \times K_0''(k\rho_0)+2\sum_{m=1}^\infty g_m(k)K_m'(k\rho_0)K_m''(k\rho_0)] \\ & +\frac{\mu_0(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0})^2}{\pi^2\rho_0^3}\int_0^\infty dk\sum_{m=1}^\infty m^2 g_m(k) \\ & \times K_m(k\rho_0)[K_m(k\rho_0)-k\rho_0K_m'(k\rho_0)] \\ & -\frac{\mu_0(\boldsymbol{m}_0\cdot\boldsymbol{e}_z)^2}{2\pi^2}\int_0^\infty dk\,k^3[g_0(k)K_0(k\rho_0)K_0'(k\rho_0) \\ & +2\sum_{m=1}^\infty g_m(k)K_m(k\rho_0)K_m'(k\rho_0)], \end{aligned}\tag{27a}$$

$$\begin{aligned}F_\phi = {} & -\frac{\mu_0(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0})(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0})}{2\pi^2\rho_0}\int_0^\infty dk\,k^2[g_0(k)K_1^2(k\rho_0) \\ & +2\sum_{m=1}^\infty g_m(k)K_{m-1}(k\rho_0)K_{m+1}(k\rho_0)], \end{aligned}\tag{27b}$$

$$F_z=0.\tag{27c}$$

Some conclusions can be drawn from the result. First, the levitation force contains in general an angular component as well as a radial one. The angular component vanishes when $(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0})(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0})=0$, that is, when $\boldsymbol{m}_0\parallel\boldsymbol{e}_{\rho 0}$ or $\boldsymbol{m}_0\perp\boldsymbol{e}_{\rho 0}$. Second, there is no simple relation among the forces for dipoles with the three independent directions, $\boldsymbol{e}_{\rho 0}$, $\boldsymbol{e}_{\phi 0}$, and $\boldsymbol{e}_z$.

There is another commonly accepted approach to the levitation force. In this approach one first calculate the self-energy of the magnetic dipole by the formula

$$U(\boldsymbol{r}_0)=-\tfrac{1}{2}\boldsymbol{m}_0\cdot\boldsymbol{B}'(\boldsymbol{r}_0),\tag{28}$$

and then obtain the levitation force through the relation

$$\boldsymbol{F}=-\nabla_0U(\boldsymbol{r}_0),\tag{29}$$

where ∇_0 is the gradient with respect to $\boldsymbol{r}_0$. It can be shown that

$$\begin{aligned}U(\boldsymbol{r}_0) = {} & \frac{\mu_0(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\rho 0})^2}{4\pi^2}\int_0^\infty dk\,k^2 \\ & \times\left[g_0(k)K_1^2(k\rho_0)+2\sum_{m=1}^\infty g_m(k)K_m'^2(k\rho_0)\right] \\ & +\frac{\mu_0(\boldsymbol{m}_0\cdot\boldsymbol{e}_{\phi 0})^2}{2\pi^2\rho_0^2}\int_0^\infty dk\sum_{m=1}^\infty m^2g_m(k)K_m^2(k\rho_0) \\ & +\frac{\mu_0(\boldsymbol{m}_0\cdot\boldsymbol{e}_z)^2}{4\pi^2}\int_0^\infty dk\,k^2\left[g_0(k)K_0^2(k\rho_0)\right. \\ & \left.+2\sum_{m=1}^\infty g_m(k)K_m^2(k\rho_0)\right]. \end{aligned}\tag{30}$$

Using the relations $\partial\boldsymbol{e}_{\rho 0}/\partial\phi_0=\boldsymbol{e}_{\phi 0}$, $\partial\boldsymbol{e}_{\phi 0}/\partial\phi_0=-\boldsymbol{e}_{\rho 0}$, it can be shown that equation (29) yields the same result as above. In this calculation it is important to place the dipole at an arbitrary position. If it is placed at some special position with, say, $\phi_0=0$ (this is equivalent to a special choice of the x axis), then the angular component of the levitation force cannot be obtained in this way.

For a magnetic point dipole in the presence of a superconducting sphere, only special cases were studied previously, where the magnetic dipole points in the radial or transverse direction [9–14]. In those cases the levitation force always points in the radial direction. If the magnetic moment of the dipole has both radial and transverse components, it may be expected that the levitation force also contains a transverse component. This should be verified by further study.

It should be pointed out that the field obtained above is only approximately valid if the source dipole is realized by a uniformly magnetized sphere, since then it is not really a point source, and its magnetization may be changed by the interaction with the superconducting cylinder. Moreover, the levitation force holds approximately only when the radius of the magnetized sphere is much smaller than ρ_0 so that the induced magnetic induction $\boldsymbol{B}'$ can be treated as uniform in the region occupied by the sphere. These remarks also apply to similar situations in the following.

4 Problems in two dimensions

In this section we will consider problems where the magnetic sources are uniform in the axial direction. In these cases all the fields and the supercurrent are independent of z, so we will use two-dimensional language and employ the polar coordinates (ρ,ϕ).

4.1 Elementary solution in two dimensions

In this section we consider the simplest case where the source is a magnetically charged long thread (or a line charge) parallel to the axis of the cylinder. In the two-dimensional language it is called a monopole, and its charge is the linear charge density q of the long thread. The line charge passes through the xy plane at the point $\boldsymbol{\rho}_0 = (\rho_0, \phi_0)$ (in the two-dimensional cases the position is always given in polar coordinates) where $\rho_0 > a$. Equivalently we will say that the monopole is located at $\boldsymbol{\rho}_0$. The solution in this case is the two-dimensional elementary solution, since solutions for more complicated charge distributions can be obtained from it by superposition.

The field equations in this case can be solved directly and it is easier than the case of Section 2. As before, φ' satisfies the Laplace equation and the two-dimensional solution can be easily written down. It is physically obvious that $B_{2z} = 0$, then it follows from Ampere's law that $\boldsymbol{J}_t = 0$. The equation for J_z is the same as that in equation (8), but the solution in two dimensions is simpler. Then $\boldsymbol{B}_{2t}$ can be easily obtained from the London equation (the second in Eq. (3b)). The source field $\boldsymbol{B}_0$ can be written down directly. By using the boundary condition (4) all unknown coefficients in the solution can be determined. However, we will find the solution by superposition as we have done in Section 3. This is another typical application of the elementary solution obtained in Section 2.

We use the same notations for the fields and supercurrent as in three dimensions. Consider the magnetic scalar potential in equation (12b). We replace Q in this equation by $q\,dz_0$ and integrate z_0 from $-\infty$ to $+\infty$. The result is the magnetic scalar potential for the two-dimensional monopole:

$$\varphi_0(\boldsymbol{\rho}) = -\frac{q}{2\pi}\ln\frac{|k|}{2}\bigg|_{k\to 0} + \frac{q}{2\pi}\Bigg[-\ln\rho_0 + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho}{\rho_0}\right)^m \cos m(\phi-\phi_0)\Bigg], \quad \rho < \rho_0. \tag{31a}$$

The first term is a divergent constant, which can be removed by choosing an appropriate reference point. Since the choice of a reference point does not affect the magnetic induction, we will drop this term directly. Then it can also be written as

$$\varphi_0(\boldsymbol{\rho}) = -\frac{q}{2\pi}\ln|\boldsymbol{\rho}-\boldsymbol{\rho}_0|, \tag{31b}$$

which is a familiar result, and is valid for $\rho > \rho_0$ as well. We then apply the same procedure to the solution given in equations (13) to (15). The results are the required solution for the current case. No divergent constant appear in these results. For the induced field we obtain

$$\varphi'(\boldsymbol{\rho}) = \frac{q}{2\pi}\sum_{m=1}^{\infty}\frac{I_{m+1}(\kappa a)}{m I_{m-1}(\kappa a)}\frac{(a^2/\rho_0)^m}{\rho^m} \times \cos m(\phi-\phi_0), \quad \rho \ge a. \tag{32a}$$

In the ideal Meissner limit $\kappa\to\infty$ this reduces to

$$\varphi'(\boldsymbol{\rho}) = \frac{q}{2\pi}\ln\rho - \frac{q}{2\pi}\ln\left|\boldsymbol{\rho} - \frac{a^2}{\rho_0^2}\boldsymbol{\rho}_0\right|, \quad \rho \ge a, \tag{32b}$$

which is the result obtained in reference [13], and is the basis of the image method for a long superconducting cylinder in the ideal Meissner state. For the magnetic induction inside the cylinder we have the nonvanishing components

$$B_{2\rho}(\boldsymbol{\rho}) = -\frac{\mu_0 q}{\pi}\sum_{m=1}^{\infty} m\frac{a^{m-1}}{\rho_0^m I_{m-1}(\kappa a)} \times \frac{I_m(\kappa\rho)}{\kappa\rho}\cos m(\phi-\phi_0), \quad \rho \le a, \tag{33a}$$

$$B_{2\phi}(\boldsymbol{\rho}) = \frac{\mu_0 q}{\pi}\sum_{m=1}^{\infty}\frac{a^{m-1}}{\rho_0^m I_{m-1}(\kappa a)} \times I'_m(\kappa\rho)\sin m(\phi-\phi_0), \quad \rho \le a. \tag{33b}$$

For the supercurrent we have only one nonvanishing component

$$J_z(\boldsymbol{\rho}) = \frac{\kappa q}{\pi}\sum_{m=1}^{\infty}\frac{a^{m-1}}{\rho_0^m I_{m-1}(\kappa a)} \times I_m(\kappa\rho)\sin m(\phi-\phi_0),\ \rho \le a. \tag{34}$$

In the ideal Meissner limit $\kappa\to\infty$ we have $B_{2\phi} = 0$ and $J_z = 0$ for $\rho < a$, and $B_{2\rho} = 0$ for $\rho \le a$, as expected.

The levitation force acted on the monopole (actually it is the force per unit length on the line charge) is

$$\boldsymbol{F} = q\boldsymbol{B}'(\boldsymbol{\rho}_0) = \boldsymbol{e}_{\rho 0}\frac{\mu_0 q^2}{2\pi}\sum_{m=1}^{\infty}\frac{I_{m+1}(\kappa a)}{I_{m-1}(\kappa a)}\frac{a^{2m}}{\rho_0^{2m+1}}. \tag{35}$$

This can be recast in the form

$$\boldsymbol{F} = \frac{\mu_0 q^2}{2\pi}\frac{a^2\boldsymbol{\rho}_0}{\rho_0^2(\rho_0^2-a^2)} - \boldsymbol{e}_{\rho 0}\frac{\mu_0 q^2}{\pi}\sum_{m=1}^{\infty}\frac{m I_m(\kappa a)}{\kappa a I_{m-1}(\kappa a)}\frac{a^{2m}}{\rho_0^{2m+1}}, \tag{36}$$

where the first term is the result obtained in the ideal Meissner limit [13], and the second term is a correction due to the finite penetration depth, which vanishes in the limit $\kappa\to\infty$. We see that the levitation force is reduced by the correction.

4.2 Magnetic point dipole in two dimensions

In this section we consider a two-dimensional magnetic point dipole with dipole moment $\boldsymbol{m}_0$ located at $\boldsymbol{\rho}_0$ (where $\rho_0 > a$). This is of physical interest since it can be realized by a long cylinder of permeable material uniformly magnetized in the transverse direction. The solution in this case can be obtained from the two-dimensional elementary solution by superposition.

The above point dipole can be theoretically realized by two monopoles, one with charge $-q$ located at $\boldsymbol{\rho}_0$ and the other with charge q located at $\boldsymbol{\rho}_0 + \boldsymbol{l}$ (here $\boldsymbol{l}$ is a two-dimensional vector), where $\boldsymbol{l} \to 0$ and $q \to \infty$ but $q\boldsymbol{l} = \boldsymbol{m}_0$ is fixed. Since $\boldsymbol{l} \to 0$ the polar coordinates of $\boldsymbol{\rho}_0 + \boldsymbol{l}$ is $(\rho_0 + \delta\rho, \phi_0 + \delta\phi)$ where $\delta\rho = \boldsymbol{l} \cdot \boldsymbol{e}_{\rho 0}$, $\delta\phi = \boldsymbol{l} \cdot \boldsymbol{e}_{\phi 0}/\rho_0$. The magnetic scalar potential of the source dipole is

$$\varphi_0(\boldsymbol{\rho}) = \frac{\boldsymbol{m}_0 \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_0)}{2\pi|\boldsymbol{\rho} - \boldsymbol{\rho}_0|^2}. \tag{37}$$

This results from the superposition of two monopole potentials of the form (31b). By similar superposition of two elementary solutions of Section 4.1, we obtain the solution for the present case. For the induced field we obtain

$$\varphi'(\boldsymbol{\rho}) = \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{I_{m+1}(\kappa a)}{I_{m-1}(\kappa a)} \frac{a^{2m}}{\rho_0^{m+1} \rho^m} \times [-(\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\rho 0}) \cos m(\phi - \phi_0) + (\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\phi 0}) \sin m(\phi - \phi_0)], \quad \rho \geq a. \tag{38a}$$

In the ideal Meissner limit this reduces to

$$\varphi'(\boldsymbol{\rho}) = \frac{\boldsymbol{m}' \cdot [\boldsymbol{\rho} - (a^2/\rho_0^2)\boldsymbol{\rho}_0]}{2\pi|\boldsymbol{\rho} - (a^2/\rho_0^2)\boldsymbol{\rho}_0|^2}, \quad \rho \geq a, \tag{38b}$$

where

$$\boldsymbol{m}' = \frac{a^2}{\rho_0^2}[\boldsymbol{m}_0 - 2(\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\rho 0})\boldsymbol{e}_{\rho 0}]. \tag{38c}$$

This is the result obtained by the image method [13]. For the magnetic induction inside the cylinder we have the nonvanishing components

$$B_{2\rho}(\boldsymbol{\rho}) = \frac{\mu_0}{\pi} \sum_{m=1}^{\infty} \frac{m^2 a^{m-1}}{\rho_0^{m+1} I_{m-1}(\kappa a)} \times \frac{I_m(\kappa\rho)}{\kappa\rho}[(\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\rho 0}) \cos m(\phi - \phi_0) - (\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\phi 0}) \sin m(\phi - \phi_0)], \quad \rho \leq a, \tag{39a}$$

$$B_{2\phi}(\boldsymbol{\rho}) = -\frac{\mu_0}{\pi} \sum_{m=1}^{\infty} \frac{m a^{m-1}}{\rho_0^{m+1} I_{m-1}(\kappa a)} \times I_m'(\kappa\rho)[(\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\rho 0}) \sin m(\phi - \phi_0) + (\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\phi 0}) \cos m(\phi - \phi_0)], \quad \rho \leq a. \tag{39b}$$

For the supercurrent we have only one nonvanishing component

$$J_z(\boldsymbol{\rho}) = -\frac{\kappa}{\pi} \sum_{m=1}^{\infty} \frac{m a^{m-1}}{\rho_0^{m+1} I_{m-1}(\kappa a)} \times I_m(\kappa\rho)[(\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\rho 0}) \sin m(\phi - \phi_0) + (\boldsymbol{m}_0 \cdot \boldsymbol{e}_{\phi 0}) \cos m(\phi - \phi_0)], \quad \rho \leq a. \tag{40}$$

In the ideal Meissner limit we have $B_{2\phi} = 0$ and $J_z = 0$ for $\rho < a$, and $B_{2\rho} = 0$ for $\rho \leq a$, as expected.

The levitation force can be found to be

$$\boldsymbol{F} = \nabla(\boldsymbol{m}_0 \cdot \boldsymbol{B}')|_{\boldsymbol{\rho}=\boldsymbol{\rho}_0} = \boldsymbol{e}_{\rho 0} \frac{\mu_0 \boldsymbol{m}_0^2}{2\pi} \sum_{m=1}^{\infty} \frac{m(m+1)I_{m+1}(\kappa a)}{I_{m-1}(\kappa a)} \frac{a^{2m}}{\rho_0^{2m+3}}. \tag{41}$$

By calculating the self-energy of the dipole and then taking the negative gradient as in Section 3, we obtain the same result. As was found in the ideal Meissner limit, the levitation force does not depend on the orientation of the dipole. This is rather different from the three-dimensional result in Section 3. The above result can be recast in the form

$$\boldsymbol{F} = \frac{\mu_0 \boldsymbol{m}_0^2}{\pi} \frac{a^2 \boldsymbol{\rho}_0}{(\rho_0^2 - a^2)^3} - \boldsymbol{e}_{\rho 0} \frac{\mu_0 \boldsymbol{m}_0^2}{\pi} \sum_{m=1}^{\infty} \frac{m^2(m+1)I_m(\kappa a)}{\kappa a I_{m-1}(\kappa a)} \frac{a^{2m}}{\rho_0^{2m+3}}, \tag{42}$$

where the first term is the result obtained in the ideal Meissner limit [13], and the second term is a correction due to the finite penetration depth, which vanishes in the limit $\kappa \to \infty$. We see again that the levitation force is reduced by the correction.

4.3 Uniform magnetic field

As another application of the two-dimensional elementary solution, we consider the influence of the superconducting cylinder on a uniform magnetic field $\boldsymbol{H}_0 = H_0 \boldsymbol{e}_x$ where H_0 is a constant.

The uniform magnetic field can be realized by two large monopoles at infinity. We put one monopole with charge $-q$ at $\boldsymbol{\rho}_0 = (L, 0)$ and the other with charge q at $\boldsymbol{\rho}_0' = (L, \pi)$. According to equation (31b) the source potential is found to be

$$\varphi_0(\boldsymbol{\rho}) = -\frac{q}{\pi L} \sum_{k=0}^{\infty} \frac{\rho^{2k+1}}{(2k+1)L^{2k}} \cos(2k+1)\phi, \quad \rho < L. \tag{43}$$

By superposition of two elementary solutions we can write down all the fields and the supercurrent for the present situation. In particular, we find that the induced magnetic induction $\boldsymbol{B}'$ is the same at $\boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_0'$. Therefore the levitation force on the monopole system is zero, so is the one on the superconducting cylinder. The conclusion also holds in the following limit case.

We only write down the results in the limit where $q, L \to \infty$ but $q/\pi L = H_0$ is fixed. In this limit only the term with $k = 0$ in the above equation contributes and we have for any finite ρ

$$\varphi_0(\boldsymbol{\rho}) = -H_0 \rho \cos\phi = -H_0 x, \quad \boldsymbol{H}_0 = H_0 \boldsymbol{e}_x. \tag{44}$$

Thus the source monopoles generate a uniform magnetic field. In this limit the fields and supercurrent are listed below. Only nonvanishing components are given.

$$\varphi'(\boldsymbol{\rho}) = -H_0 a^2 \frac{I_2(\kappa a)}{I_0(\kappa a)} \frac{\cos\phi}{\rho^2}, \quad \rho \geq a. \tag{45}$$

$$B_{2\rho}(\boldsymbol{\rho}) = \frac{2\mu_0 H_0}{I_0(\kappa a)} \frac{I_1(\kappa\rho)}{\kappa\rho} \cos\phi,$$

$$B_{2\phi}(\boldsymbol{\rho}) = -\frac{2\mu_0 H_0}{I_0(\kappa a)} I_1'(\kappa\rho) \sin\phi, \quad \rho \leq a. \tag{46}$$

$$J_z(\boldsymbol{\rho}) = -\frac{2\kappa H_0}{I_0(\kappa a)} I_1(\kappa\rho) \sin\phi, \quad \rho \leq a. \tag{47}$$

In the ideal Meissner limit φ' reduces to the result obtained by the image method [13], while $B_{2\phi} = 0$ and $J_z = 0$ for $\rho < a$, and $B_{2\rho} = 0$ for $\rho \leq a$. These are all expected results.

4.4 Steady electric current

In this section we consider a long straight wire parallel to the z axis carrying an electric current I. It passes through the xy plane at the point $\boldsymbol{\rho}_0$ (where $\rho_0 > a$ as before). The magnetic induction for this current is

$$\boldsymbol{B}_0(\boldsymbol{\rho}) = \frac{\mu_0 I}{2\pi} \frac{\boldsymbol{e}_z \times (\boldsymbol{\rho} - \boldsymbol{\rho}_0)}{|\boldsymbol{\rho} - \boldsymbol{\rho}_0|^2}. \tag{48}$$

Since the electric current is not a superposition of magnetic charges, the solution for the present case cannot be obtained from the elementary solution. Therefore the problem should be solved individually. The approach to the solution has been outlined in the second paragraph of Section 4.1. We only give the results here. In order to determine the coefficients in the solution, the above source field should be expanded as

$$B_{0\rho}(\boldsymbol{\rho}) = -\frac{\mu_0 I}{2\pi} \sum_{m=1}^{\infty} \frac{\rho^{m-1}}{\rho_0^m} \sin m(\phi - \phi_0),$$

$$B_{0\phi}(\boldsymbol{\rho}) = -\frac{\mu_0 I}{2\pi} \sum_{m=1}^{\infty} \frac{\rho^{m-1}}{\rho_0^m} \cos m(\phi - \phi_0), \quad \rho < \rho_0. \tag{49}$$

For the induced field we obtain

$$\varphi'(\boldsymbol{\rho}) = \frac{I}{2\pi} \sum_{m=1}^{\infty} \frac{I_{m+1}(\kappa a)}{m I_{m-1}(\kappa a)} \frac{(a^2/\rho_0)^m}{\rho^m} \sin m(\phi - \phi_0), \ \rho \geq a. \tag{50a}$$

In the ideal Meissner limit the corresponding magnetic induction can be recast in the form

$$B'(\boldsymbol{\rho}) = \frac{\mu_0 I}{2\pi\rho} \boldsymbol{e}_\phi - \frac{\mu_0 I}{2\pi} \frac{\boldsymbol{e}_z \times [\boldsymbol{\rho} - (a^2/\rho_0^2)\boldsymbol{\rho}_0]}{|\boldsymbol{\rho} - (a^2/\rho_0^2)\boldsymbol{\rho}_0|^2}, \quad \rho \geq a, \tag{50b}$$

which is the result obtained by the image method [13]. For the magnetic induction inside the cylinder we have the nonvanishing components

$$B_{2\rho}(\boldsymbol{\rho}) = -\frac{\mu_0 I}{\pi} \sum_{m=1}^{\infty} \frac{m a^{m-1}}{\rho_0^m I_{m-1}(\kappa a)} \times \frac{I_m(\kappa\rho)}{\kappa\rho} \sin m(\phi - \phi_0), \quad \rho \leq a, \tag{51a}$$

$$B_{2\phi}(\boldsymbol{\rho}) = -\frac{\mu_0 I}{\pi} \sum_{m=1}^{\infty} \frac{a^{m-1}}{\rho_0^m I_{m-1}(\kappa a)} \times I_m'(\kappa\rho) \cos m(\phi - \phi_0), \quad \rho \leq a. \tag{51b}$$

For the supercurrent we have only one nonvanishing component

$$J_z(\boldsymbol{\rho}) = -\frac{\kappa I}{\pi} \sum_{m=1}^{\infty} \frac{a^{m-1}}{\rho_0^m I_{m-1}(\kappa a)} I_m(\kappa\rho) \cos m(\phi - \phi_0), \ \rho \leq a. \tag{52}$$

In the ideal Meissner limit we have $B_{2\phi} = 0$ and $J_z = 0$ for $\rho < a$, and $B_{2\rho} = 0$ for $\rho \leq a$, as expected. One can also confirm the above results by examining the limit $\kappa \to 0$ or $a \to 0$.

The levitation force (per unit length) on the source current is

$$\boldsymbol{F} = \boldsymbol{e}_{\rho 0} \frac{\mu_0 I^2}{2\pi} \sum_{m=1}^{\infty} \frac{I_{m+1}(\kappa a)}{I_{m-1}(\kappa a)} \frac{a^{2m}}{\rho_0^{2m+1}}. \tag{53}$$

This has essentially the same form as the levitation force for the two-dimensional monopole obtained in equation (35), thus the subsequent discussions below that equation also apply here.

5 The mixed state

In this section we briefly discuss the case when the cylinder is in the mixed state so that there exist vortex lines or rings inside it. We only discuss the effect of these vortices on the magnetic source outside the cylinder. We do not consider the influence of the magnetic source on the creation and distribution of the vortices. It is obvious that the solution to the field equations when there exist both external sources and vortex lines can be obtained by superposition of the solution with only external sources and the one with only vortex lines. Therefore it is sufficient to consider only vortex lines in this section. Furthermore, the solution when there are several vortex lines are the superposition of the several solutions each for one individual vortex line. First we consider a vortex line parallel to the axis of the cylinder and centered at the position $\boldsymbol{\rho}_1$ (here $\rho_1 < a$). The field equations for this case have been solved in the literature [15]. It turns out that the field is confined to the cylinder. Therefore one or more such vortex lines have no effect on the magnetic source outside the cylinder, and the levitation force on the source is the same as that obtained in the case when the cylinder is in the Meissner state. Second we consider a vortex ring concentric with the cylinder, which has the radius ρ_1 (again $\rho_1 < a$) and vertical position z_1. By solving the field equations it turns out again that the fields outside the cylinder is null [16]. Therefore vortex rings of this type have no effect on the magnetic source outside the cylinder either. Third we may consider vortex lines along some diameter of the cylinder, or even more general ones. It may be expected that such vortex lines would have effect on the magnetic source outside the cylinder. However, it is somewhat difficult to solve the field equations in this case, and the situation needs further study.

6 Summary and discussions

In this paper a general formalism for solving the Maxwell–London equations for a general magnetic source in the presence of a long superconducting cylinder in the Meissner state is studied. By this formalism we first solve the simplest case when the source is a magnetic monopole. The result is then used as the elementary solution for the subsequent cases where the source is a magnetic point dipole with arbitrary direction, a two-dimensional monopole, a two-dimensional point dipole and a uniform magnetic field perpendicular to the axis of the cylinder. The solutions in all these cases are obtained from the elementary solution by superposition. In principle, this can be used for an arbitrary distribution of magnetic charges. We also solve separately the case where the source field is generated by a current carrying long straight wire parallel to the axis of the cylinder. The levitation force between the superconducting cylinder and the magnetic source is calculated in all cases. It turns out that the levitation force on a point dipole contains in general an angular component as well as a radial one. Similar result may be expected for a superconducting sphere. On the other hand, for a two-dimensional point dipole the levitation force always points in the radial direction and its magnitude is also independent of the dipole's orientation. The ideal Meissner limit is discussed in all cases and the two-dimensional results are compared with those obtained by the image method. We also discussed the mixed state of the superconducting cylinder. It was already known that vortex lines parallel to the axis of the cylinder and vortex rings concentric with the cylinder have no field outside the cylinder. Therefore the levitation forces remain the same as those obtained for the Meissner state. Nontrivial effect may be expected for a vortex line along some diameter of the cylinder, or more general ones. This should be examined by further investigation.

The formalism studied in this paper can also be used to solve some other situations that are not discussed here. For example, the case with a current carrying circular loop concentric with the cylinder may be solved without difficulty. Based on the symmetry it may be expected that the levitation force in this case is zero. On the other hand, a nontrivial result may be expected for a non-concentric one. However, this is rather difficult mathematically.

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References

1. E.H. Brandt, Appl. Phys. Lett. **53**, 1554 (1988); E.H. Brandt, Science **243**, 349 (1989); E.H. Brandt, Am. J. Phys. **58**, 43 (1990)
2. J.H. Xu, J.H. Miller, Jr, C.S. Ting, Phys. Rev. B **51**, 424 (1995); **55**, 11877 (E) (1997); A. Badía, Phys. Rev. B **55**, 11875 (1997)
3. J. Pearl, Appl. Phys. Lett. **5**, 65 (1964); J. Pearl, J. Appl. Phys. **37**, 4139 (1966)
4. Z.J. Yang, Jpn J. Appl. Phys. **31**, L477 (1992); ibid. **31**, L938 (1992); Physica C **234**, 263 (1994)
5. W.M. Saslow, Am. J. Phys. **59**, 16 (1991)
6. M.W. Coffey, Phys. Rev. B **52**, R9851 (1995)
7. M.W. Coffey, Phys. Rev. B **52**, 1187 (1995)
8. M.W. Coffey, E.T. Phipps, Phys. Rev. B **53**, 389 (1996)
9. Z.J. Yang, Solid State Commun. **107**, 745 (1998)
10. M.W. Coffey, J. Supercond. **13**, 381 (2000)
11. M.W. Coffey, Phys. Rev. B **65**, 214524 (2002)
12. M.W. Coffey, J. Supercond. **15**, 257 (2002)
13. Q.-G. Lin, Phys. Rev. B **74**, 024510 (2006)
14. Q.-G. Lin, Comment on *"London model for the levitation force between a horizontally oriented point magnetic dipole and superconducting sphere",* Phys. Rev. B (in press)
15. P.A. Venegas, E. Sardella, Phys. Rev. B **58**, 5789 (1998)
16. Y.A. Genenko, Phys. Rev. B **49**, 6950 (1994)
17. A. Sanchez, C. Navau, Phys. Rev. B **64**, 214506 (2001); C. Navau, A. Sanchez, Phys. Rev. B **64**, 214507 (2001)
18. J.D. Jackson, *Classical Electrodynamics*, 3rd edn. (Wiley, New York, 1998)
19. F. Hellman, E.M. Gyorgy, D.W. Johnson, Jr. H.M. O'Bryan, R. C. Sherwood, J. Appl. Phys. **63**, 447 (1988)
20. Z.J. Yang, T.H. Johansen, H. Bratsberg, G. Helgesen, A.T. Skjeltorp, Physica C **160**, 461 (1989)
21. L.C. Davis, J. Appl. Phys. **67**, 2631 (1990)
22. R.J. Adler, W.W. Anderson, J. Appl. Phys. **68**, 695 (1990)
23. T. Torng, Q.Y. Chen, J. Appl. Phys. **73**, 1198 (1993)
24. M.J. Qin, G. Li, H.K. Liu, S.X. Dou, E.H. Brandt, Phys. Rev. B **66**, 024516 (2002)
25. C. Navau, A. Sanchez, Phys. Rev. B **58**, 963 (1998)
26. Y. Martin, H.K. Wickramasinghe, Appl. Phys. Lett. **50**, 1455 (1987)
27. J.J. Sáenz, N. García, P. Grütter, E. Meyer, H. Heinzelmann, R. Wiesendanger, L. Rosenthaler, H.R. Hidber, H.-J. Güntherodt, J. Appl. Phys. **62**, 4293 (1987)
28. D. Rugar, H.J. Mamin, P. Güthner, S.E. Lambert, J.E. Stern, I. McFadyen, T. Yogi, J. Appl. Phys. **68**, 1169 (1990)
29. H.J. Reittu, R. Laiho, Supercond. Sci. Technol. **5**, 448 (1992)